Student Number:

Set:

Shore

Year 12
Mathematics Extension 1
Trial HSC Examination
2011

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total Marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

(a) The interval AB, where $A\ (1,\ -2)$ and $B\ (6,\ 7)$, is divided externally in the ratio 2 : 3 by the point $P\ (x,\ y)$. Find the values of x and y. **2**

(b) Find the domain and range of the function $f(x) = 2\sin^{-1}3x$. **2**

(c) Solve $x-1\le\dfrac{2}{x}$. **3**

(d) Differentiate $\sin^{-1}(\cos x)$. Write your answer in simplest form. **2**

(e) Use the substitution $u=1+x$ to find $\displaystyle\int\frac{x}{\sqrt{1+x}}\,dx$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The acceleration of a particle is given by $\ddot{x} = 4(x+1)$ ms^{-2}. Initially, the particle is at the origin and velocity is 2 ms^{-1}.

(i) Show that the velocity, v, at any position, x, is given by $v = 2x + 2$. **2**

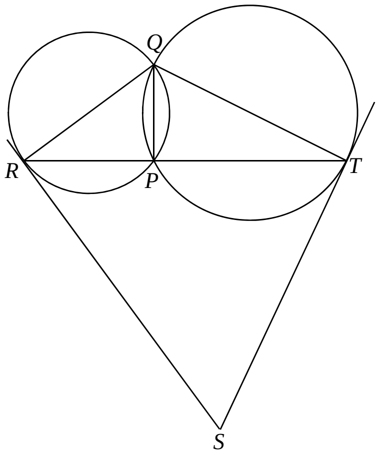
(ii) Hence show that $x = e^{2t} - 1$. **2**

(b)

(i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$ where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $\cos\theta - \sqrt{3}\sin\theta = 1$ for $0 \le \theta \le 2\pi$. **2**

(c)



NOT TO SCALE

Two circles intersect at P and Q. RPT is a straight line where R is a point on the first circle and T is a point on the second circle. The tangent at R and the tangent at T meet at S.

Copy the diagram into your booklet.

Prove that $QRST$ is a cyclic quadrilateral. **4**

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)

(i) Show that the equation $e^{-x} = \sin 2x$ has a root lying between 1 and 2. **1**

(ii) By taking 1.5 as a first approximation, use Newton's method once, to obtain a better approximation to this root. (Give your answer correct to 2 decimal places). **2**

(b) Let T be the temperature in a room at time t and let A be the temperature of its surroundings. Newton's Law of Cooling states that the rate of change of temperature T is proportional to $(T - A)$ i.e. $\dfrac{dT}{dt} = k(T - A)$.

(i) Verify that $T = A + Be^{kt}$ is a solution to $\dfrac{dT}{dt} = k(T - A)$. **1**

(ii) If the temperature of a substance in a room of constant temperature $6°\text{C}$ is noted to be $29°\text{C}$ and in 40 minutes to be $14°\text{C}$, find the value of B and k. **2**

(iii) Find how long it takes for the temperature of the substance to reach $9°\text{C}$. Give your answer to the nearest minute. **2**

(c) The polynomial $P(x) = 2x^3 + kx^2 + 3x - 4$ has roots α, β and γ.

(i) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(ii) If one root is the reciprocal of the other, find the third root and hence find the value of k. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

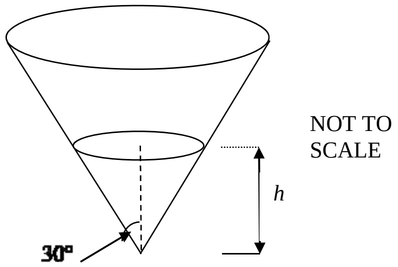
(a) Evaluate $\int_0^3 \frac{dx}{9+x^2}$. **2**

(b) Find the term independent of *x* in the expansion $\left(x^2+\frac{2}{x}\right)^9$. **3**

(c) The region bounded by the curve $y=3\sin 2x$, the *x*–axis and the line $x=\frac{\pi}{4}$ is rotated about the *x*-axis to form a solid of revolution. **3**

Find the volume of the solid formed.

(d)



A hollow cone with vertical angle $60°$ is held with its axis vertical and vertex downwards.

Sand is being poured into the cone at a uniform rate of 15 cubic metres per second.

(i) Show that when the sand level has reached a height of *h* metres, the volume of the sand in the cone, in cubic metres, is given by $V=\frac{1}{9}\pi h^3$. **2**

(ii) Find the rate at which the sand level is rising when its depth is 4 metres. (Express your answer in terms of π). **2**

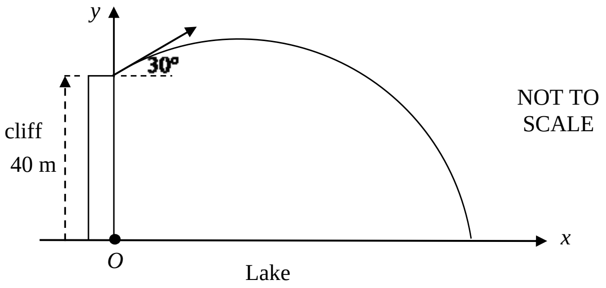
Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the function $f(x)=e^{x+2}$

(i) Find the inverse function $f^{-1}(x)$. **2**

(ii) Sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$ clearly indicating any intercepts with the axes. **2**

(b)



A pebble is projected from the top of a vertical cliff with velocity 20 ms^{-1} at an angle of elevation of $30°$. The cliff is 40 metres high and overlooks a lake.

Assume that, *t* seconds after release of the pebble, the horizontal displacement from *O* is given by $x=10\sqrt{3}t$. (DO NOT prove this).

(i) Assuming the acceleration due to gravity is 10 ms^{-2}, show that the pebble's vertical displacement is given by $y=-5t^2+10t+40$. **2**

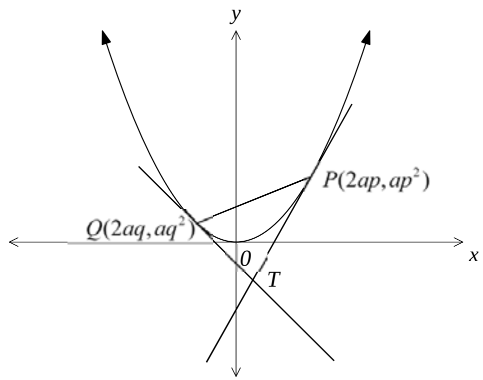
(ii) Calculate the time which elapses before the pebble hits the lake and the distance of the point of impact from the foot of the cliff. **3**

(iii) Find the angle and the speed at which the pebble hits the lake. **3**

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Use the process of mathematical induction to prove that $2^{3n}-1$ is divisible by 7 for all integers $n \geq 1$. **3**

(b)



NOT TO SCALE

In the diagram above a focal chord PQ intersects the parabola $x^2 = 4ay$ at points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. The tangents to the parabola at point P and point Q intersect at T.

(i) Show that the equation of the chord PQ is given by $y = \left(\frac{p+q}{2}\right)x - apq$. **2**

(ii) Show that $pq = -1$. **1**

(iii) Show that the acute angle between the focal chord QP and the tangent TP to the parabola at P is given by $\tan^{-1}|q|$. **3**

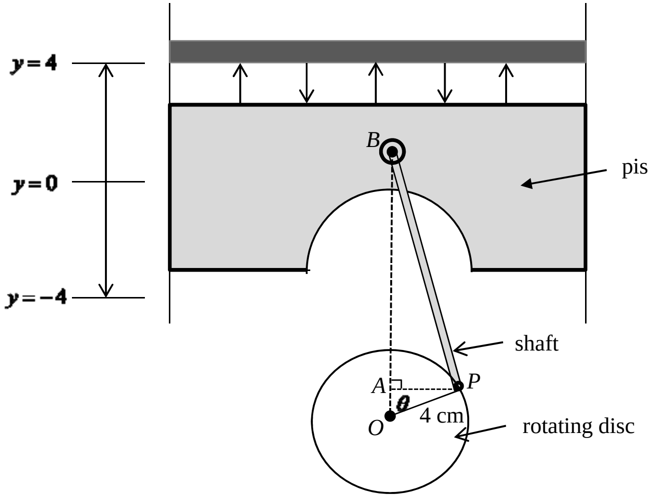
(c) Consider the expansion of $\left(\frac{1}{3}+2x\right)^{18}$.

(i) Show that the $(k+1)$th term is given by $T_{k+1} = {}^{18}C_k \frac{2^k}{3^{18-k}} x^k$. **1**

(ii) Hence, or otherwise, find the greatest coefficient of the expansion $\left(\frac{1}{3}+2x\right)^{18}$. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

The diagram below illustrates the movement of a piston head in a combustion engine. As the disc rotates, the shaft (BP) moves the piston head up and down. The radius of the rotating disc is 4 cm and the vertical displacement of the piston at any time t seconds is given by the equation $y = 4\cos 10t$.



NOT TO SCALE

(i) Find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ and hence show that the piston moves in simple harmonic motion. **2**

(ii) State the period of the motion in exact form. **1**

(iii) Find the exact velocity of the piston when $y = 3$ cm at the first time it reaches this point. **2**

(iv) Show that $OB = 4\cos\theta + \sqrt{BP^2 - 16\sin^2\theta}$ **3**

Question 7 continued over the page……..

(v) If the disc is rotating at a rate of 10 radians/second, and **3**

$$\frac{d(OB)}{dt}=\frac{d(OB)}{d\theta}\times\frac{d\theta}{dt} \text{ show that}$$

$$\frac{d(OB)}{dt}=-40\sin\theta\left[1+\frac{\cos\theta}{\sqrt{\left(\frac{BP}{4}\right)^2-\sin^2\theta}}\right]$$

(vi) Calculate $\frac{d(OB)}{dt}$, the rate at which the distance OB is changing over time, when $\theta=\pi$. **1**

End of Paper

2011 EXTENSION 1 TRIAL HSC

Question 1:

a) $(1,-2) \quad (6,7)$

$-2 : 3$

$x = \dfrac{mx_2 + nx_1}{m+n} \qquad y = \dfrac{my_2 + ny_1}{m+n}$

$= \dfrac{-2\times 6 + 3\times 1}{1} \qquad = \dfrac{-2\times 7 + 3\times -2}{1}$

$= -9 \qquad = -20$

$\therefore \underline{P(-9,-20)}$ [2]

b) $f(x) = 2\sin^{-1} 3x$

Domain: $-1 \leqslant 3x \leqslant 1$

$-\frac{1}{3} \leqslant x \leqslant \frac{1}{3}$

Range: $-\frac{\pi}{2} \leqslant \frac{y}{2} \leqslant \frac{\pi}{2}$

$-\pi \leqslant y \leqslant \pi$ [2]

c) critical points

1. $x \neq 0$
2. Solve $x - 1 = \dfrac{2}{x}$

$x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = -1$ or 2

3. Test

$x = -2$ ✓ $\quad \therefore \underline{x \leqslant -1 \text{ or } 0 < x \leqslant 2}$

$x = -\frac{1}{2}$ ✗

$x = 1$ ✓

$x = 3$ ✗

[3]

d) $y = \sin^{-1}(\cos x)$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\cos^2 x}} \times -\sin x$

$= \dfrac{-\sin x}{\sqrt{\sin^2 x}}$

$= \dfrac{-\sin x}{\sin x}$

$= \underline{-1}$ [2]

e) $u = 1 + x \Rightarrow x = u - 1$

$\dfrac{du}{dx} = 1$

$du = dx$

$\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$

$\displaystyle= \int \frac{u-1}{u^{1/2}}\,du$

$\displaystyle= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$= \dfrac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + c$

$= \frac{2}{3}\sqrt{(1+x)^3} - 2\sqrt{1+x} + c$

[3]

Question 2:

a) $\quad t = 0,\ x = 0$

$\ddot{x} = 4(x+1) \quad v = 2$

$\displaystyle\int \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 4\int (x+1)\,dx$

$\frac{1}{2}v^2 = 4\left[\dfrac{x^2}{2} + x\right] + c$

$x = 0 \quad v = 2$

$2 = 4(0+0) + c$

$2 = c$

$\frac{1}{2}v^2 = 4\left(\dfrac{x^2}{2} + x\right) + 2$

$v^2 = 8\left(\dfrac{x^2}{2} + x\right) + 4$

$= 4x^2 + 8x + 4$

$= 4(x^2 + 2x + 1)$

$= 4(x+1)^2$

$v = \pm 2(x+1) \quad v > 0$

$\therefore \underline{v = 2(x+1)}$ m/s [2]

(ii) $v = 2x + 2$

$\dfrac{dx}{dt} = 2x + 2$

$\dfrac{dt}{dx} = \dfrac{1}{2(x+1)}$

$t = \dfrac{1}{2}\displaystyle\int \frac{1}{x+1}\,dx$

$t = \frac{1}{2}\ln(x+1) + c$

$t = 0 \quad x = 0$

$0 = \frac{1}{2}\ln 1 + c$

$0 = c$

$\therefore t = \frac{1}{2}\ln(x+1)$

$2t = \ln(x+1)$

$x + 1 = e^{2t}$

$\underline{x = e^{2t} - 1}$ which is required

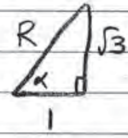
[2]

(b) $\cos\theta - \sqrt{3}\sin\theta = R\cos(\theta + \alpha)$

$R\cos\theta\cos\alpha - R\sin\theta\sin\alpha = R\cos(\theta + \alpha)$

$\therefore R\cos\alpha = 1 \qquad R\sin\alpha = \sqrt{3}$

$\cos\alpha = \dfrac{1}{R} \qquad \sin\alpha = \dfrac{\sqrt{3}}{R}$



$\therefore R^2 = 3 + 1$

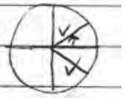
$\underline{R = 2}$

$\tan\alpha = \sqrt{3}$

$\underline{\alpha = \dfrac{\pi}{3}}$

$\therefore \cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \dfrac{\pi}{3}\right)$ [2]

(ii) $2\cos\left(\theta + \dfrac{\pi}{3}\right) = 1$



$\cos\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{1}{2} \qquad \angle = \dfrac{\pi}{3}$

for $0 \leq \theta \leq 2\pi$

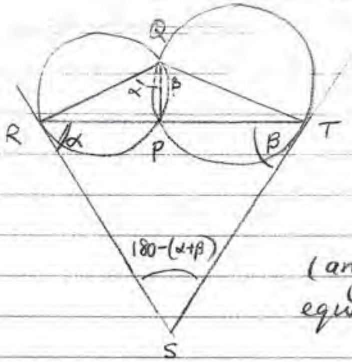
$\dfrac{\pi}{3} \leq \theta + \dfrac{\pi}{3} \leq \dfrac{7\pi}{3}$

$\therefore \theta + \dfrac{\pi}{3} = \dfrac{\pi}{3}, \dfrac{7\pi}{3}$ or $\theta + \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

$\theta = 0, 2\pi$, or $\dfrac{4\pi}{3}$

ie $\theta = 0, \dfrac{4\pi}{3}$ or 2π [2]

(c)



$\angle SRT = \angle RQP = \alpha$

(angle between a chord & a tangent is equal to the ∠ in the alternate segm

$\angle STP = \angle TQP = \beta$

(angle be-tween a chord & tangent is equal to the ∠ in the alternate segment

Now $\angle RQT = \alpha+\beta$ (by addition)

In ΔSRT, $\angle RST = 180 - (\alpha+\beta)$ (Angle sum of ΔRTS)

$\angle RQT + \angle RST = (\alpha+\beta) + 180 - (\alpha+\beta)$

$= 180°$

∴ ∠RQT and ∠RST are supplementary ∠s.

Hence RQTS is a cyclic quadrilateral. [4]

Question 3:

2)

(i) let $f(x) = e^{-x} - \sin 2x$

$f(1) = e^{-1} - \sin 2 = -0.54\ldots$

$f(2) = e^{-2} - \sin 4 = 0.89\ldots$

since $f(1) < 0$ and $f(2) > 0$ then there is a root between 1 and 2.

[1]

(ii) $f'(x) = -e^{-x} - 2\cos 2x$

$f'(1.5) = -e^{-1.5} - 2\cos 3$

$= 1.7568.$

$f(1.5) = e^{-1.5} - \sin 3$

$= 0.082$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1.5 - \dfrac{0.082}{1.7568}$

$= 1.4533\ldots$

$= \underline{\underline{1.45}}$ (2dp) [2]

b) (i) $T = A + Be^{kt}$ (1)

$\dfrac{dT}{dt} = Bke^{kt}$ (2)

but from (1) $T - A = Be^{kt}$

∴ sub into (2)

$\dfrac{dT}{dt} = K(T-A)$ ∴ a soln.

[1]

(ii) $t=0$ $T=29$ $t=40$ $T=14°$

$A = 6°$

When $t=0$, $T=29$

$29 = 6 + Be^{0}$

$\underline{23 = B}$

$T = 6 + 23e^{kt}$

$14 = 6 + 23e^{40k}$

$\dfrac{8}{23} = e^{40k}$

$\ln\left(\dfrac{8}{23}\right) = 40k$

$\dfrac{1}{40}\ln\left(\dfrac{8}{23}\right) = k$

[2]

(iii) $t = ?$ $T = 9$

$9 = 6 + 23e^{kt}$

$\dfrac{3}{23} = e^{kt}$

$\ln\left(\dfrac{3}{23}\right) = kt$

$\dfrac{1}{k}\ln\left(\dfrac{3}{23}\right) = t$

$t = 77.15$ [2]

$t \doteqdot$ <u>77 minutes</u> (nearest minute)

(c) $P(x) = 2x^3 + kx^2 + 3x - 4$

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \dfrac{3/2}{-4/2}$

$= \underline{\underline{\dfrac{3}{4}}}$

[2]

(ii) Let the roots be $\alpha, \frac{1}{\alpha}, \beta$

$\alpha \times \frac{1}{\alpha} \times \beta = 2 \Rightarrow \underline{\underline{\beta = 2}}$

$P(2) = 16 + 4k + 6 - 4 = 0$

$4k = -18$ [2]

$\underline{k = -4\frac{1}{2}}$

Question 4:

a) $\displaystyle\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$

$= \frac{1}{3}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$

$= \underline{\underline{\dfrac{\pi}{12}}}$ [2]

b) ${}^9C_k\,(x^2)^{9-k}\,(2x^{-1})^k$

$= {}^9C_k\,2^k . x^{18-2k} . x^{-k}$

$= {}^9C_k\,2^k . x^{18-3k}$

Term independent of x is x^0

∴ $x^0 = x^{18-3k}$

$0 = 18 - 3k$

$\underline{k = 6}$

∴ Term is ${}^9C_6\,2^6 = \underline{\underline{5376}}$ [3]

c) $V = \pi \int_0^{\pi/4} (9\sin^2 2x)\,dx$

$= 9\pi \int_0^{\pi/4} \sin^2 2x \; dx$

$\left.\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ 2\sin^2 x &= 1 - \cos 2x \\ \sin^2 x &= \tfrac{1}{2}(1-\cos 2x) \\ \sin^2 2x &= \tfrac{1}{2}(1-\cos 4x) \end{aligned}\right\}$

$\therefore \; V = \frac{9\pi}{2} \int_0^{\pi/4} (1-\cos 4x)\,dx$

$= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/4}$

$= \frac{9\pi}{2}\left[\left(\frac{\pi}{4} - \frac{1}{4}\sin\pi\right)\right.$

$\left. - \left(0 - \frac{1}{4}\sin 0\right)\right]$

$= \frac{9\pi}{2}\left[\frac{\pi}{4} - 0\right]$

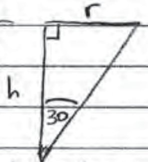
$= \frac{9\pi^2}{8}$ units^3. [3]

d) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$= \frac{1}{3}\pi \frac{h^2}{3} h$

$= \frac{1}{9}\pi h^3$



$\tan 30^\circ = \frac{r}{h}$

$r = h\tan 30^\circ$

$= \frac{h}{\sqrt{3}}$

[2]

(ii) given $\frac{dV}{dt} = 15$

$V = \frac{1}{9}\pi h^3$

$\frac{dV}{dh} = \frac{1}{3}\pi h^2$

$\therefore \; \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{3}{\pi h^2} \times 15$

$= \frac{45}{\pi h^2}$

When $h = 4$ $\quad \frac{dh}{dt} = \frac{45}{16\pi}$ m/s [2]

Question 5:

a) $f(x) = e^{x+2}$ $\quad \left[\begin{array}{l} D: \text{all real } x \\ R: \; y > 0 \end{array}\right.$

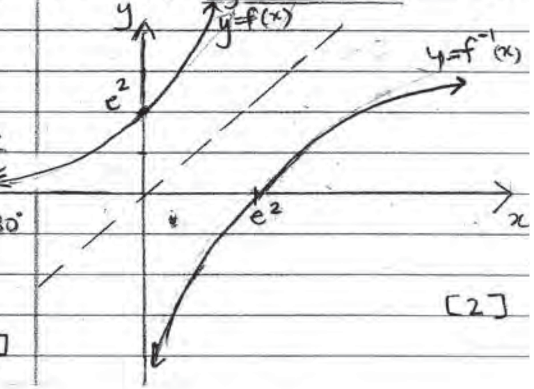
(i) $y = e^{x+2}$

$x = e^{y+2}$

$\ln x = y + 2$

$\ln x - 2 = y$

$\therefore f^{-1}(x)$: $y = \ln x - 2$ [2]



[2]

b) $\ddot{y} = -10$

$\dot{y} = -10t + c_1$

When $t=0$ $\quad \dot{y} = 10$

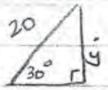
$\therefore c_1 = 10$

$\dot{y} = -10t + 10$

$y = -5t^2 + 10t + c_2$

$t=0$ $\quad y = 40 \; \therefore \; c_2 = 40$

$y = -5t^2 + 10t + 40$ [2]



$\left[\begin{array}{l} \sin 30^\circ = \frac{\dot{y}}{20} \\ \\ \dot{y} = 20 \times \frac{1}{2} \\ \dot{y} = 10 \end{array}\right.$

(ii) Hits the lake when $y = 0$

$5t^2 - 10t - 40 = 0$

$t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$t = 4$ or -2 $\quad$ but $t > 0$

$\therefore t = 4$ s

When $t = 4$ $\quad x = 10\sqrt{3} \times 4$

$x = 40\sqrt{3}$ m [3]

(iii) $x = 10\sqrt{3}t$

$\dot{x} = 10\sqrt{3}$

At $t = 4$ $\quad \dot{y} = -40 + 10$

$\dot{y} = -30$

$\dot{v}^2 = \dot{x}^2 + \dot{y}^2$

$= (10\sqrt{3})^2 + (-30)^2$

$V = \sqrt{1200}$ $\quad V > 0$

Speed $= 20\sqrt{3}$ m/s

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right| =$

$= \left|\frac{-30}{10\sqrt{3}}\right|$

$\tan\theta = \frac{3}{\sqrt{3}}$

$\theta = 60^\circ$ [3]

Question 6:

Step 1: Prove true for $n=1$

$2^3 - 1 = 7$ which is divisible by 7.

step 2: Assume true for $n=k$

ie $2^{3k} - 1 = 7M$ where M is an integer

$2^{3k} = 7M + 1$ $\quad$ (i)

Now, prove true for $n = k+1$

$2^{3(k+1)} - 1 = 2^{3k+3} - 1$

$= 2^{3k} \cdot 2^3 - 1$

$= (7M+1)8 - 1$ $\quad$ From (i)

$= 56M + 8 - 1$

$= 56M + 7$

$= 7(8M+1)$

which is divisible by 7

step 3: since true for $n=1$, then true for $n = 1+1 = 2$, $n=3$ and so on for all integers $n \geq 1$.

[3]

(b) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

(i) $m_{PQ} = \frac{aq^2 - ap^2}{2aq - 2ap}$

$= \frac{\cancel{a}\cancel{(q-p)}(q+p)}{2\cancel{a}\cancel{(q-p)}}$

$= \frac{p+q}{2}$

$\therefore$ Eqn PQ

$y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$

$y - ap^2 = \left(\frac{p+q}{2}\right)x - ap^2 - apq$

$y = \left(\frac{p+q}{2}\right)x - apq$ [2]

(ii) since a focal chord, passes thro $(0,a)$

$a = 0 - apq$

$\frac{a}{-a} = pq$

$pq = -1$ [1]

(iii) $m_1 = \frac{p+q}{2}$ $m_2 = p$

$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{p+q}{2} - p}{1 + \left(\frac{p+q}{2}\right)p}\right|$

$= \left|\frac{\frac{p+q-2p}{2}}{\frac{2+p^2+pq}{2}}\right|$

$= \left|\frac{p+q-2p}{2+p^2+pq}\right|$

$= \left|\frac{q-p}{p^2+pq+2}\right|$

Now $pq = -1$

$p = -\frac{1}{q}$

$= \left|\frac{q + \frac{1}{q}}{\frac{1}{q^2} - 1 + 2}\right|$

$= \left|\frac{\frac{q^2+1}{q}}{\frac{1}{q^2}+1}\right|$

$= \left|\frac{\frac{q^2+1}{q}}{\frac{q^2+1}{q^2}}\right|$

$= |q|$

$\theta = \tan^{-1}|q|$ [3]

(c) $\left(\frac{1}{3} + 2x\right)^{18}$

(i) $T_{k+1} = {}^{n}C_k\, a^{n-k}\, b^k$

$= {}^{18}C_k \left(\frac{1}{3}\right)^{18-k} (2x)^k$

$= {}^{18}C_k \frac{1}{3^{18-k}} 2^k \cdot x^k$

$= {}^{18}C_k \frac{2^k}{3^{18-k}} x^k$ [1]

(ii) $\frac{T_{k+1}}{T_1} = \frac{{}^{18}C_k \frac{2^k}{3^{18-k}}}{{}^{18}C_{k-1} \frac{2^{k-1}}{3^{18-k+1}}}$

$= \frac{\cancel{18!}}{\underset{k}{\cancel{k!}}\,\cancel{(18-k)!}} \times \frac{\cancel{(k-1)!}\,\overset{18-k+1}{\cancel{(18-k+1)!}}}{\cancel{18!}} \times 2^{k-k+1} \times 3^{18-k+1-18+k}$

$= \frac{19-k}{k} \cdot 2 \cdot 3$

$= \frac{114-6k}{k}$

Greatest coeff when $\frac{T_{k+1}}{T_k} \geq 1$

$114 - 6k \geq k$

$114 \geq 7k$

$16\frac{2}{7} \geq k$

$\therefore$ For $k = 1, 2, 3, \ldots 16$ $T_{k+1} > T_k$ and for $k = 17, 18, \ldots$ $T_{k+1} < T_k$

$\therefore$ Greatest coeff when $k = 16$

Greatest Coefficient is ${}^{18}C_{16} \frac{2^{16}}{3^2} = 1\,114\,112.$ [2]

(i) $y = 4\cos 10t$

$\frac{dy}{dt} = -40\sin 10t$

$\frac{d^2y}{dt^2} = -400\cos 10t$

but $y = 4\cos 10t$

$\therefore \frac{d^2y}{dt^2} = -100\,(4\cos 10t)$

$= -100y.$ $\quad (n = 10)$

$\therefore$ since piston is moving in form $\ddot{y} = -ny$ then moving in SHM

[2]

(ii) $T = \frac{2\pi}{n}$

$= \frac{2\pi}{10}$

$= \frac{\pi}{5}$ $\quad$ [1]

(iii) $y = 3$ $\quad 3 = 4\cos 10t$

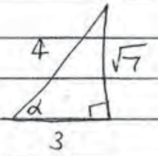
$\frac{3}{4} = \cos 10t$

$\cos^{-1}\frac{3}{4} = 10t$

$\frac{1}{10}\cos^{-1}\frac{3}{4} = t$

$\dot{y} = -40\sin 10\left(\frac{1}{10}\cos^{-1}\frac{3}{4}\right)$

$= -40\sin\left(\cos^{-1}\frac{3}{4}\right)$



$= -40 \times \frac{\sqrt{7}}{4}$ $\quad \sin\alpha = \frac{\sqrt{7}}{4}$

$= \underline{\underline{-10\sqrt{7}}}$ cm/s $\quad$ [2]

(iv) $OB = OA + AB$

In Δ OAP, $\cos\theta = \frac{OA}{4}$ $\qquad \sin\theta = \frac{AP}{4}$

$OA = 4\cos\theta$ ① $\qquad AP = 4\sin\theta$

In ΔABP, by pythagoras) theorem

$BP^2 = AP^2 + AB^2$

$BP^2 = (4\sin\theta)^2 + AB^2$

$BP^2 - 16\sin^2\theta = AB^2$

$\therefore AB = \sqrt{BP^2 - 16\sin^2\theta}$ $\quad AB > 0$ ② $\qquad$ [3]

(v) $\left(\frac{d\theta}{dt} = 10\right)$ $\therefore OB = 4\cos\theta + \sqrt{BP^2 - 16\sin^2\theta}.$

$OB = 4\cos\theta + (BP^2 - 16\sin^2\theta)^{\frac{1}{2}}$

$\frac{d(OB)}{d\theta} = -4\sin\theta + \frac{1}{2}(BP^2 - 16\sin^2\theta)^{-\frac{1}{2}} \times -32\sin\theta\cos\theta$

$= -4\sin\theta - \frac{16\sin\theta\cos\theta}{\sqrt{BP^2 - 16\sin^2\theta}}$

$= -4\sin\theta - \frac{16\sin\theta\cos\theta}{\sqrt{16\left(\left(\frac{BP}{4}\right)^2 - \sin^2\theta\right.}}$

$= -4\sin\theta - \frac{16\sin\theta\cos\theta}{4\sqrt{\left(\frac{BP}{4}\right)^2 - \sin^2\theta}}$

$= -4\sin\theta - \frac{4\sin\theta\cos\theta}{\sqrt{\left(\frac{BP}{4}\right)^2 - \sin^2\theta}}$

$= -4\sin\theta\left[1 + \frac{\cos\theta}{\sqrt{\left(\frac{BP}{4}\right)^2 - \sin^2\theta}}\right]$ $\qquad$ [3]

Now given $\frac{d(OB)}{dt} = \frac{d(OB)}{d\theta}) \times \frac{d\theta}{dt}$

$= -4\sin\theta\left[1 + \frac{\cos\theta}{\sqrt{(BP/4)^2 - \sin^2\theta}}\right] \times 10$

$$= -40\sin\theta\left[1 + \frac{\cos\theta}{\sqrt{\left(\frac{BP}{4}\right)^2 - \sin^2\theta}}\right]$$ which is required.

(vi) when $\theta = 90^\circ$

$$\frac{d(OB)}{dt} = -40\sin 90^\circ\left[1 + \frac{\cos 90^\circ}{\sqrt{\left(\frac{BP}{4}\right)^2 - \sin^2(90^\circ)}}\right]$$

$\sin 90^\circ = 1 \quad \cos 90^\circ = 0$

$$\therefore \frac{d(OB)}{dt} = -40\,(1 + 0)$$

$$= \underline{\underline{-40\ \text{cm/s}}}$$ [1]